ONA GENERALIZED DIFFERENCE DOUBLE SEQUENCE SPACES DEFINED BY SEQUENCE OF ORLICZ FUNCTION

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Abstract :

The idea of difference sequence spaces was introduced by kizmaz [4], and this concept was generalized by Malkowsky [8]. Recently, Asma [1] defined the sequence spaces as $c_0(u, \Delta, M, p)$, $c(u, \Delta, M, p)$, and $l_\infty(u, \Delta, M, p)$

Throughout this study ω is a family of all $\mathbb{C}^2$ double sequence (i.e. $x = x_{k,l}$ and $y = y_{k,l}$ are complex double sequence), w denote the family of all $\mathbb{C}$ double sequences , and $2l_\infty, 2c$ and $2c_\circ$ be the linear spaces of bounded, convergent, and null sequences with complex terms, respectively, normed by

$$\|(x, y)\| = sup_{k,l}(|x_{k,l}|, |y_{k,l}|)$$

Where $\|x\| = sup_{k,l}|x_{k,l}|$, $\|y\| = sup_{k,l}|y_{k,l}|$, $k, l \in \mathbb{N}$, the set of positive integers .

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Definition and notation :

In this work we define the double sequence spaces $2c_0(\Delta_u^n, M_{k,l}, p, s)$, $2c(\Delta_u^n, M_{k,l}, p, s)$ and $2l_\infty(\Delta_u^n, M_{k,l}, p, s)$, where $M(x, y) = (M_1(x), M_2(y)) = \left(M_{1_{k,l}}(x), M_{2_{k,l}}(y)\right) = M_{k,l}(x, y)$ where $M_1 = M_{1_{k,l}}, M_2 = M_{2_{k,l}}$ and $M = M_{k,l}$ are a double sequences an Orlicz function, and examine some inclusion relation and properties of these spaces which will give as a special case the spaces $c_0(u, \Delta, M, p)$, $c(u, \Delta, M, p)$ and $l_\infty(u, \Delta, M, p)$ of Asma [ ].

The convergence of double sequence we mean the convergence on the Pringsheim sense that is,

Definition. A double sequence $x = x_{k,l}$, $y = y_{k,l}$ has Pringsheim limit ℓ_1, ℓ_2 denoted by ($p - \lim x = \ell_1$, $p - \lim y = \ell_2$) provided that given $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_{k,l} - \ell_1| < \epsilon$, $|y_{k,l} - \ell_2| < \epsilon$ implies that $|(x_{k,l} - \ell_1, y_{k,l} - \ell_2)| < (\epsilon, \epsilon)$ whenever $k, l > N$. We shall describe such an $x = x_{k,l}$, $y = y_{k,l}$ more briefly as "P –convergent". We shall denote the space of all P –convergent sequences by $2c$.

Defintion. The double sequence $x = x_{k,l}$, $y = y_{k,l}$ is bounded if and only if there exists a positive number M such that $x_{k,l} < M$, $y_{k,l} < M$ we have $(x_{k,l}, y_{k,l}) < (M, M)$ for all k, l, i.e. if $\|x\|_\infty = sup_{k,l}|x_{k,l}|$, $\|y\|_\infty = sup_{k,l}|y_{k,l}|$ implies that $\|(x,y)\|_{(\infty,\infty)} = sup_{k,l}(|x_{k,l}|, |y_{k,l}|)$ We shall denote all bounded double sequences by $2l_\infty$.

Definition. A double sequence $x = x_{k,l}$, $y = y_{k,l}$ are said to be Cauchy sequence if for every $\epsilon >$ 0ther exist $N \in \mathbb{N}$ such that $|x_{i,j} - x_{k,l}| < \epsilon$, $|y_{i,j} - y_{k,l}| < \epsilon$, impels that $|(x_{i,j} - x_{k,l}, y_{i,j} - y_{k,l})| < (\epsilon, \epsilon)$.

For any double sequence $x = x_{k,l}, y = y_{k,l}$ such that $(x, y) = (x_{k,l}, y_{k,l})$ the difference of a double sequence $(\Delta x, \Delta y)$ is define by

$$(\Delta x, \Delta y) = \left(\Delta x_{k,l}, \Delta y_{k,l}\right)_{k,l=1}^{\infty} = \left(x_{k,l} - x_{k,l+1} - x_{k+1,l} + x_{k+1,l+1}, y_{k,l} - y_{k,l+1} - y_{k+1,l} + y_{k+1,l+1}\right)_{k,l=1}^{\infty}$$

$$\Delta x = \left(\Delta x_{k,l}\right)_{k,l=1}^{\infty} = \left(x_{k,l} - x_{k,l+1} - x_{k+1,l} + x_{k+1,l+1}\right)_{k,l=1}^{\infty},$$

$$\Delta y = \left(\Delta y_{k,l}\right)_{k,l=1}^{\infty} = \left(y_{k,l} - y_{k,l+1} - y_{k+1,l} + y_{k+1,l+1}\right)_{k,l=1}^{\infty}$$

According to kizmaz sense we define the double sequence spaces as follows :

$2l_\infty(\Delta) = \{(x, y) \in \omega : (\Delta x, \Delta y) \in 2l_\infty\}$

$2c\,(\Delta) = \{(x, y) \in \omega : (\Delta x, \Delta y) \in 2c\}$

$2c_0(\Delta) = \{(x, y) \in \omega : (\Delta x, \Delta y) \in 2c_0\}$

where $2l_\infty(\Delta) = (l_\infty(\Delta x), l_\infty(\Delta y))$, $2\,c(\Delta) = (c\,(\Delta x), c\,(\Delta y))$, $2c_0(\Delta) = (c_0(\Delta x), c_0(\Delta y))$

Let U be the set of all double sequence $u = u_{k,l}$ such that $u_{k,l} \neq 0$ $(k = 1,2, \dots)$, $(l = 1,2, \dots)$

We define a double sequence spaces in Malkowsky sense as follows:

$2l_\infty(u, \Delta) = \{(x, y) \in \omega : (\Delta x, \Delta y) \in 2l_\infty\}$

$2c\,(u, \Delta) = \{(x, y) \in \omega : (\Delta x, \Delta y) \in 2c\}$

$2c_0(u, \Delta) = \{(x, y) \in \omega : (\Delta x, \Delta y) \in 2c_0\}$

where $2l_\infty(u, \Delta) = (l_\infty(u, \Delta x), l_\infty(u, \Delta y))$, $2c(\Delta) = (c\,(u, \Delta x), c\,(u, \Delta y))$

, $2c_0(\Delta) = (c_0(u, \Delta x), c_0(u, \Delta y))$

Definition : A double Orlicz function is a function
$M: [0, \infty) \times [0, \infty) \to [0, \infty) \times [0, \infty)$ such that
$M(x, y) = (M_1(x), M_2(y))$
$M_1: [0, \infty) \to [0, \infty)$
$M_2: [0, \infty) \to [0, \infty)$, where M_1, M_2 are Orlicz functions
which is continuous, non-decreasing ,even , convex and satisfies the following conditions
i) $M(0,0) = (0,0)$
ii) $M_1(x) > 0, M_2(y) > 0 \Longrightarrow M(x, y) = (M_1(x), M_2(y)) > (0,0)$ for $x > 0, y > 0$ we mean by $M(x, y) > (0,0)$ that $M_1(x) > 0, M_2(y) > 0$
ii) $M_1(x) \to 0, M_2(y) \to 0$ as $x \to \infty$, $y \to \infty$, then $M(x, y) \to \infty$

Definition : An Orlicz function M is said to satisfy the $\Delta_2 -$ condition for all values of δ if there exist a constant $K > 0$ such that
$M(2\delta) \leq K\,M(\delta)$. $\delta \geq 0$
equivalently,
$M(h\,\delta) \leq K\,h\,M(\delta)$for every values of δ and for $\delta > 1$

Now ,we use the idea of Orlicz function to define what is called an Orlicz double sequence space, by means of Lindenstrauss and Tzafriri sense as follows :

$2l_M = \left\{(x,y) \in \omega : \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\left[M_1\left(\frac{|x_{k,l}|}{\rho}\right) \vee M_2\left(\frac{|y_{k,l}|}{\rho}\right)\right] < \infty,\ for\ som\ \rho > 0\right\}$

where $2l_M = \left(l_{M_1}, l_{M_2}\right)$ Such that

$l_{M_1} = \left\{x \in w : \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} M_1\left(\frac{|x_{k,l}|}{\rho}\right) < \infty,\ for\ som\ \rho > 0\right\}$

and

$l_{M_2} = \left\{y \in w : \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} M_2\left(\frac{|y_{k,l}|}{\rho}\right) < \infty,\ for\ som\ \rho > 0\right\}$

which is a Banach space with the norm :

$\| (x,y) \|_M = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} sup\left[M_1\left(\frac{|x_{k,l}|}{\rho}\right) \vee M_2\left(\frac{|y_{k,l}|}{\rho}\right)\right] \le 1\right\}$

where

$\| x \|_{M_1} = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} M_1\left(\frac{|x_{k,l}|}{\rho}\right) \le 1\right\}$

$\| y \|_{M_2} = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} M_2\left(\frac{|y_{k,l}|}{\rho}\right) \le 1\right\}$.

According to the concept of Mursuleen et al [ ],we can define The double sequence spaces as follows :

$2\, l_\infty(\Delta, M) = \left\{(x,y) \in \omega : sup_{k,l}\left[M_1\left(\frac{|x_{k,l}|}{\rho}\right) \vee M_2\left(\frac{|y_{k,l}|}{\rho}\right)\right] < \infty,\ \text{for some } \rho > 0\right\}$,

$2c\,(\Delta, M) = \left\{(x,y) \in \omega : \lim_{k,l\to\infty}\left[M_1\left(\frac{|\Delta x_{k,l} - \ell_1|}{\rho}\right) \vee M_2\left(\frac{|\Delta y_{k,l} - \ell_2|}{\rho}\right)\right] = 0,\ \text{for some } \rho > 0, \ell_1, \ell_2 \in \mathbb{C}\right\}$,

and

$2c_o(\Delta, M) = \left\{(x,y) \in \omega : \lim_{k,l\to\infty}\left[M_1\left(\frac{|\Delta x_{k,l}|}{\rho}\right) \vee M_2\left(\frac{|\Delta y_{k,l}|}{\rho}\right)\right] = 0,\ \text{for some } \rho > 0\right\}$,

where $M = \left(M_1(x), M_2(y)\right)$ is an Orlicz function, and we show that these spaces are Banach spaces with the norm

$\| (x,y) \|_\Delta = inf\ \left\{\rho > 0 : sup_{k,l}\left[M_1\left(\frac{|\Delta x_{k,l}|}{\rho}\right) \vee M_2\left(\frac{|\Delta y_{k,l}|}{\rho}\right)\right] \le 1\right\}$

$\| x \|_\Delta = inf\ \left\{\rho > 0 : sup_{k,l} M_1\left(\frac{|\Delta x_{k,l}|}{\rho}\right) \le 1\right\}$

$\| y \|_\Delta = inf\ \left\{\rho > 0 : sup_{k,l} M_2\left(\frac{|\Delta y_{k,l}|}{\rho}\right) \le 1\right\}$

In Asma sense , also we can define and study the double sequence spaces as follows :

$2l_\infty(u, \Delta, M) = \left\{(x,y) \in \omega : sup_{k,l}\ \left[M_1\left(\frac{|u_{k,l}\,\Delta x_{k,l}|}{\rho}\right) \vee M_2\left(\frac{|u_{k,l}\,\Delta y_{k,l}|}{\rho}\right)\right] < \infty,\ \text{for some } \rho > 0\right\}$,

$2c(u, \Delta, M) = \left\{(x,y) \in \omega : \lim_{k,l\to\infty}\left[M_1\left(\frac{|u_{k,l}\,\Delta x_{k,l} - \ell_1|}{\rho}\right) \vee M_2\left(\frac{|u_{k,l}\,\Delta y_{k,l} - \ell_2|}{\rho}\right)\right] = 0,\ \text{for some } \rho > 0, \ell_1, \ell_2 \in \mathbb{C}\right\}$,

$$2c_0(u,\Delta,M) = \left\{ (x,y) \in \omega : \lim_{k,l\to\infty} \left[M_1\left(\frac{|u_{k,l}\,\Delta x_{k,l}|}{\rho}\right) \vee M_2\left(\frac{|u_{k,l}\,\Delta y_{k,l}|}{\rho}\right)\right] = 0, \text{ for some } \rho > 0 \right\}.$$

In this study we further generalize these spaces as follows :

Let $M_1 = M_{1_{k,l}}$, $M_2 = M_{2_{k,l}}$ and $M = M_{k,l} = \left(M_{1_{k,l}}, M_{2_{k,l}}\right)$ be a double sequence of Orlicz functions and n be a positive integer, so, we use the notation $\Delta_u^n x_{k,l}$ for $u_{k,l}\Delta^n x_{k,l}$, $\Delta_u^n y_{k,l}$ for $u_{k,l}\Delta^n y_{k,l}$such that $\left(\Delta_u^n x_{k,l}, \Delta_u^n y_{k,l}\right)$ for $\left(u_{k,l}\Delta^n x_{k,l}\right.$ $\left., u_{k,l}\Delta^n y_{k,l}\right)$,we define

$$2c_0\left(\Delta_u^n, M_{k,l}, s\right) = \left\{(x,y) \in \omega : \lim_{k,l\to\infty} (kl)^{-s} \left[M_{1_{k,l}}\left(\frac{|\Delta_u^n x_{k,l}|}{\rho}\right) \vee M_{2_{k,l}}\left(\frac{|\Delta_u^n y_{k,l}|}{\rho}\right)\right] = 0, \text{ for some } \rho > 0, s \geq 0\right\},$$

$$2c\left(\Delta_u^n, M_{k,l}, s\right) = \left\{(x,y) \in \omega : \lim_{k,l\to\infty} (kl)^{-s} \left[M_{1_{k,l}}\left(\frac{|\Delta_u^n x_{k,l} - \ell_1|}{\rho}\right) \vee M_{2_{k,l}}\left(\frac{|\Delta_u^n y_{k,l} - \ell_2|}{\rho}\right)\right] = 0,\right.$$
$$\left.\text{for some } \rho > 0, \ell_1, \ell_2 \in \mathbb{C}, s \geq 0\right\},$$

and

$$2l_\infty\left(\Delta_u^n, M_{k,l}, s\right) = \left\{ (x,y) \in \omega : sup_{k,l}\ (kl)^{-s} \left[M_{1_{k,l}}\left(\frac{|\Delta_u^n x_{k,l}|}{\rho}\right) \vee M_{2_{k,l}}\left(\frac{|\Delta_u^n y_{k,l}|}{\rho}\right)\right] < \infty,\right.$$
$$\left.\text{for some } \rho > 0, s \geq 0\right\}.$$

where

$\Delta_u^n x_{k,l} = \Delta_u^{n-1} x_{k,l} - \Delta_u^{n-1} x_{k,l+1} - \Delta_u^{n-1} x_{k+1,l} + \Delta_u^{n-1} x_{k+1,l+1}$, and

$\Delta_u^n y_{k,l} = \Delta_u^{n-1} y_{k,l} - \Delta_u^{n-1} y_{k,l+1} - \Delta_u^{n-1} y_{k+1,l} + \Delta_u^{n-1} y_{k+1,l+1}$

We have $\left(\Delta_u^n x_{k,l}, \Delta_u^n y_{k,l}\right) = \left(\Delta_u^{n-1} x_{k,l} - \Delta_u^{n-1} x_{k,l+1} - \Delta_u^{n-1} x_{k+1,l} + \Delta_u^{n-1} x_{k+1,l+1},\ \Delta_u^{n-1} y_{k,l} - \Delta_u^{n-1} y_{k,l+1} - \Delta_u^{n-1} y_{k+1,l} + \Delta_u^{n-1} y_{k+1,l+1}\right)$

such that

$$\Delta_u^n x_{k,l} = \sum_{r_1=0}^n \sum_{r_2=0}^n (-1)^{r_1+r_2} \binom{n}{r_1}\binom{n}{r_2} u_{k+r_1,l+r_2} x_{k+r_1,l+r_2} \text{ ,and}$$

$$\Delta_u^n y_{k,l} = \sum_{r_1=0}^n \sum_{r_2=0}^n (-1)^{r_1+r_2} \binom{n}{r_1}\binom{n}{r_2} u_{k+r_1,l+r_2} y_{k+r_1,l+r_2}$$

We have $\left(\Delta_u^n x_{k,l}, \Delta_u^n y_{k,l}\right) = \left(\sum_{r_1=0}^n \sum_{r_2=0}^n (-1)^{r_1+r_2} \binom{n}{r_1}\binom{n}{r_2} u_{k+r_1,l+r_2} x_{k+r_1,l+r_2},\ \sum_{r_1=0}^n \sum_{r_2=0}^n (-1)^{r_1+r_2} \binom{n}{r_1}\binom{n}{r_2} u_{k+r_1,l+r_2} y_{k+r_1,l+r_2}\right)$

$\Delta_u^0 x_{k,l} = u_{k,l} x_{k,l}$, $\Delta_u^0 y_{k,l} = u_{k,l} y_{k,l}$,

hence $\left(\Delta_u^0 x_{k,l},\ \Delta_u^0 y_{k,l}\right) = \left(u_{k,l} x_{k,l}, u_{k,l} y_{k,l}\right)$

$\Delta_u x_{k,l} = u_{k,l} x_{k,l} - u_{k,l+1} x_{k,l+1} - u_{k+1,l} x_{k+1,l} + u_{k+1,l+1} x_{k+1,l+1}$

and

$\Delta_u y_{k,l} = u_{k,l} y_{k,l} - u_{k,l+1} y_{k,l+1} - u_{k+1,l} y_{k+1,l} + u_{k+1,l+1} y_{k+1,l+1}$

implies that

$(\Delta_u x_{k,l}, \Delta_u y_{k,l}) = (u_{k,l}x_{k,l} - u_{k,l+1}x_{k,l+1} - u_{k+1,l}x_{k+1,l} + u_{k+1,l+1}x_{k+1,l+1}, u_{k,l}y_{k,l} - u_{k,l+1}y_{k,l+1} - u_{k+1,l}y_{k+1,l} + u_{k+1,l+1}y_{k+1,l+1})$

If $M_{1_{k,l}}, M_{2_{k,l}} = M_1, M_2$ respectively for all k,l and $s = 0, n = 1$, then these spaces reduce to those of Asma sense [1]

Main results 2.2:

We prove the following theorems :

Theorem 2.1.2.1. $2l_\infty(\Delta_u^n, M_{k,l}, \mathrm{s})$ is a Banach space with the norm

$$\|(x,y)\|_{\Delta_u^n} = inf\left\{ \rho > 0: sup_{k,l}(kl)^{-s}\left[M_{1_{k,l}}\left(\frac{|\Delta_u^n x_{k,l}|}{\rho}\right) \vee M_{2_{k,l}}\left(\frac{|\Delta_u^n y_{k,l}|}{\rho}\right)\right] \leq 1\right\}.$$

where

$$\|x\|_{\Delta_u^n} = inf\left\{ \rho > 0 : sup_{k,l}(kl)^{-s}M_{1_{k,l}}\left(\frac{|\Delta_u^n x_{k,l}|}{\rho}\right) \leq 1\right\}$$

$$\|y\|_{\Delta_u^n} = inf\left\{ \rho > 0 : sup_{k,l}(kl)^{-s}M_{2_{k,l}}\left(\frac{|\Delta_u^n y_{k,l}|}{\rho}\right) \leq 1\right\}$$

Proof. Let (x^i, y^i) be any double Cauchy sequence in $2l_\infty(\Delta_u^n, M_{k,l}, \mathrm{s})$, such that x^i and y^i be a Cauchy sequence in $l_\infty(\Delta_u^n, M_{1_{k,l}}, \mathrm{s})$, $l_\infty(\Delta_u^n, M_{2_{k,l}}, \mathrm{s})$ respectively ,where $(x^i, y^i) = (x^i_{k,l}, y^i_{k,l}) = \left((x^i_{1,1}, y^i_{1,1}), (x^i_{2,2}, y^i_{2,2}), \ldots\right) \in 2l_\infty(\Delta_u^n, M_{k,l}, \mathrm{s})$, for each $i \in \mathbb{N}$. Let $r, x_0, y_0 > 0$ be fixed, Then for each $\frac{\epsilon}{r\,x_0} > 0$,$\frac{\epsilon}{r\,y_0} > 0$ we have $\left(\frac{\epsilon}{r\,x_0}, \frac{\epsilon}{r\,y_0}\right) > (0,0)$ there exists a positive integers L such that

$x^i - x^j{}_{\Delta_u^n} < \frac{\epsilon}{rx_0}$, $y^i - y^j{}_{\Delta_u^n} < \frac{\epsilon}{ry_0}$, we have

$(x^i - x^j\Delta_u^n, y^i - y^j\Delta_u^n) < \left(\frac{\epsilon}{r\,x_0}, \frac{\epsilon}{r\,y_0}\right)$ for all $i, j \geq L$.

Using the definition of norm , we have

$$Sup_{k,l}(kl)^{-s}\left[M_{1_{k,l}}\left(\frac{\left|\Delta_u^n x^i_{k,l} - \Delta_u^n x^j_{k,l}\right|}{x^i - x^j_{\Delta_u^n}}\right) \vee M_{2_{k,l}}\left(\frac{\left|\Delta_u^n y^i_{k,l} - \Delta_u^n y^j_{k,l}\right|}{y^i - y^j_{\Delta_u^n}}\right)\right] \leq 1, \text{ for all } i, j \geq L$$

Thus

$$(kl)^{-s}\left[M_{1_{k,l}}\left(\frac{\left|\Delta_u^n x^i_{k,l} - \Delta_u^n x^j_{k,l}\right|}{x^i - x^j_{\Delta_u^n}}\right) \vee M_{2_{k,l}}\left(\frac{\left|\Delta_u^n y^i_{k,l} - \Delta_u^n y^j_{k,l}\right|}{y^i - y^j_{\Delta_u^n}}\right)\right] \leq 1$$

, for all $k,l \geq 0$, for all $k,l \geq 0$, and for all $i, j \geq L$.

Therefore one can find that there exists $r > 0$ with

$(kl)^{-s}\left[M_{1_{k,l}}\left(\frac{r\,x_0}{2}\right) \vee M_{2_{k,l}}\left(\frac{r\,y_0}{2}\right)\right] \geq 1$, such that

$$(kl)^{-s}\left[M_{1_{k,l}}\left(\frac{\left|\Delta_u^n x^i_{k,l} - \Delta_u^n x^j_{k,l}\right|}{x^i - x^j_{\Delta_u^n}}\right) \vee M_{2_{k,l}}\left(\frac{\left|\Delta_u^n y^i_{k,l} - \Delta_u^n y^j_{k,l}\right|}{y^i - y^j_{\Delta_u^n}}\right)\right] \leq (kl)^{-s}\left[M_{1_{k,l}}\left(\frac{rx_0}{2}\right) \vee M_{2_{k,l}}\left(\frac{ry_0}{2}\right)\right].$$

This implies that

$\left|\Delta_u^n x^i_{k,l} - \Delta_u^n x^j_{k,l}\right| \leq \frac{r\,x_0}{2} \cdot \frac{\epsilon}{r\,x_0} = \frac{\epsilon}{2}, \left|\Delta_u^n y^i_{k,l} - \Delta_u^n y^j_{k,l}\right| \leq \frac{r\,y_0}{2} \cdot \frac{\epsilon}{ry_0} = \frac{\epsilon}{2}$ we have

$$\left|\left(\Delta_u^n x^i_{k,l} - \Delta_u^n x^j_{k,l}, \Delta_u^n y^i_{k,l} - \Delta_u^n y^j_{k,l}\right)\right| \leq \left(\frac{r\,x_0}{2} \cdot \frac{\epsilon}{r\,x_0}, \frac{r\,y_0}{2} \cdot \frac{\epsilon}{r\,y_0}\right) = \left(\frac{\epsilon}{2}, \frac{\epsilon}{2}\right)$$

Since $u_{k,l} \neq 0$ for all k, l, we get that $\left|\Delta^n x^i_{k,l} - \Delta^n x^j_{k,l}\right| \leq \frac{\epsilon}{2}$ *and* $\left|\Delta^n y^i_{k,l} - \Delta^n y^j_{k,l}\right| \leq \frac{\epsilon}{2}$ we have $\left|\left(\Delta^n x^i_{k,l} - \Delta^n x^j_{k,l}, y^i_{k,l} - \Delta^n y^j_{k,l}\right)\right| \leq \left(\frac{\epsilon}{2}, \frac{\epsilon}{2}\right)$, for all $i, j \geq L$. Hence $\Delta^n x^i_{k,l}$, $\Delta^n y^i_{k,l}$ is a Cauchy sequence in $\mathbb{R}$ such that $\left(\Delta^n x^i_{k,l}, \Delta^n y^i_{k,l}\right)$ is a double Cauchy sequence in $\mathbb{R} \times \mathbb{R}$.

Therefore for each ϵ $(0 < \epsilon < 1)$, there exists a positive integer L such that $\left|\Delta^n x^i_{k,l} - \Delta^n x_{k,l}\right| < \epsilon$ and $\left|\Delta^n y^i_{k,l} - \Delta^n y_{k,l}\right| < \epsilon$ we have $\left|\left(\Delta^n x^i_{k,l} - \Delta^n x_{k,l}, \Delta^n y^i_{k,l} - \Delta^n y_{k,l}\right)\right| < (\epsilon, \epsilon)$ for all $i \geq L$.

Now, using the continuity of $M_{1_{k,l}}, M_{2_{k,l}}$ for each k, l, we get that

$$Sup_{k,l \geq L} (kl)^{-s} \left[M_{1_{k,l}} \left(\frac{\left|\Delta^n_u x^i_{k,l} - \lim_{j\to\infty} \Delta^n_u x^j_{k,l}\right|}{\rho} \right) \vee M_{2_{k,l}} \left(\frac{\left|\Delta^n_u y^i_{k,l} - \lim_{j\to\infty} \Delta^n_u y^j_{k,l}\right|}{\rho} \right) \right] \leq 1$$

Thus

$$Sup_{k,l \geq L} (kl)^{-s} \left[M_{1_{k,l}} \left(\frac{\left|\Delta^n_u x^i_{k,l} - \Delta^n_u x_{k,l}\right|}{\rho} \right) \vee M_{2_{k,l}} \left(\frac{\left|\Delta^n_u y^i_{k,l} - \Delta^n_u y_{k,l}\right|}{\rho} \right) \right] \leq 1$$

Taking infimum of such ρ's we have

$$inf \left\{ \rho > 0 : Sup_{k,l \geq L} (kl)^{-s} \left[M_{1_{k,l}} \left(\frac{\left|\Delta^n_u x^i_{k,l} - \Delta^n_u x_{k,l}\right|}{\rho} \right) \vee M_{2_{k,l}} \left(\frac{\left|\Delta^n_v y^i_{k,l} - \Delta^n_v y_{k,l}\right|}{p} \right) \right] \leq 1 \right\} \leq \epsilon \quad \text{for all}$$

$i \geq L$ and $j \to \infty$

Since $(x^i, y^i) \in 2l_\infty(\Delta^n_u, M_{k,l}, s)$, and $M_{1_{k,l}}, M_{2_{k,l}}$ be an Orlicz function ,then $M_{k,l} = \left(M_{1_{k,l}}, M_{2_{k,l}}\right)$ is an Orlicz function for each k, l and there for continuous, we get that $(x, y) \in 2l_\infty(\Delta^n_u, M_{k,l}, s)$ ◙

Theorem 2.1.2.2. Let $M_{k,l}$ be a double sequence of Orlicz functions where $M_{k,l} = \left(M_{1_{k,l}}, M_{2_{k,l}}\right)$ which means that $M_{1_{k,l}}, M_{2_{k,l}}$ a double sequence of Orlicz function so ,it satisfies the Δ_2 −condition for each k, l . Then

i) $2l_\infty(\Delta^n_u, s) \subset 2l_\infty(\Delta^n_u, M_{k,l}, s)$

ii) $2c(\Delta^n_u, s) \subset 2c(\Delta^n_u, M_{k,l}, s)$

iii) $2c_0(\Delta^n_u, s) \subset 2c_0(\Delta^n_u, M_{k,l}, s)$

Proof. i) Let $(x, y) \in 2l_\infty(\Delta^n_u, s)$, Then $\left|\left(\Delta^n_u x_{k,l}, \Delta^n_u y_{k,l}\right)\right| \leq (L_1, L_2)$ such that $\left|\Delta^n_u x_{k,l}\right| \leq L_1$ and $\left|\Delta^n_u y_{k,l}\right| \leq L_2$, for all k, l , and L_1, L_2 be a positive integer .Therefore

$(kl)^{-s} \left[M_{1_{k,l}} \left(\frac{\left|\Delta^n_u x_{k,l}\right|}{\rho} \right) \vee M_{2_{k,l}} \left(\frac{\left|\Delta^n_u y_{k,l}\right|}{\rho} \right) \right] \leq (kl)^{-s} \left[M_{1_{k,l}} \left(\frac{L_1}{\rho} \right) \vee M_{2_{k,l}} \left(\frac{L_2}{\rho} \right) \right] \leq$

$(kl)^{-s} . K. h\left[M_{1_{k,l}}(L_1) \vee M_{2_{k,l}}(L_2) \right]$, for each k, l,

by the Δ_2 −condition. Hence

$sup_{k,l} (kl)^{-s} \left[M_{1_{k,l}} \left(\frac{\left|\Delta^n_u x_{k,l}\right|}{\rho} \right) \vee M_{2_{k,l}} \left(\frac{\left|\Delta^n_u y_{k,l}\right|}{\rho} \right) \right] < \infty$That is, $2l_\infty(\Delta^n_u, s) \subset 2l_\infty(\Delta^n_u, M_{k,l}, s)$ ◙

(ii) and (iii) can be proved in a similar way .

Theorem 2.1.2.3. Let $M_{k,l}$ be a double sequence of Orlicz functions we have $M_{1_{k,l}}, M_{2_{k,l}}$ be a double sequence of Orlicz function

i) $2l_\infty(\Delta_u^0, M_{k,l}, s) \subset 2l_\infty(\Delta_u^n, M_{k,l}, s)$,

ii) $2c(\Delta_u^0, M_{k,l}, s) \subset 2c(\Delta_u^n, M_{k,l}, s)$,

iii) $2c_0(\Delta_u^0, M_{k,l}, s) \subset 2c_0(\Delta_u^n, M_{k,l}, s)$.

Proof :it is clear

2.3 Paranormed sequence spaces

Let $p = p_{k,l}$ be a double sequence of positive real numbers. Then we can define a double sequence spaces in Mursaleen ,et al sense [8] as the follows:

$$2l_\infty(\Delta, M, p) = \left\{(x,y) \in \omega : Sup_{\ k,l} \left[M_1\left(\frac{|\Delta x_{k,l}|}{\rho}\right) \vee M_2\left(\frac{|\Delta y_{k,l}|}{\rho}\right)\right]^{p_{k,l}} < \infty,\ for\ some\ \rho > 0\right\},$$

$$2c\,(\Delta, M, p) = \left\{(x,y) \in \omega : \lim_{k,l\to\infty} \left[M_1\left(\frac{|\Delta x_{k,l}-\ell_1|}{\rho}\right) \vee M_2\left(\frac{|\Delta y_{k,l}-\ell_2|}{\rho}\right)\right]^{p_{k,l}} = 0,\ for\ some\ \rho > 0, \ell_1, \ell_2 \in \mathbb{C}\right\},$$

And

$$2c_0\,(\Delta, M, p) = \left\{(x,y) \in \omega : \lim_{k,l\to\infty} \left[M_1\left(\frac{|\Delta x_{k,l}|}{\rho}\right) \vee M_2\left(\frac{|\Delta y_{k,l}|}{\rho}\right)\right]^{p_{k,l}} = 0,\ for\ some\ \rho > 0\right\}$$

Where M is an Orlicz function such that M_1, M_2 is Orlicz function , we showed that these spaces are paranormed spaces with:

$$G\big((x,y)\big) = \inf\left\{\rho^{p_m/H} > 0: \left[\left[Sup_{\ k,l}\left[M_1\left(\frac{|\Delta x_{k,l}|}{\rho}\right) \vee\ M_2\left(\frac{|\Delta y_{k,l}|}{\rho}\right)\right]\right]^{p_{k,l}}\right]^{1/H} \le 1\right\},$$

such that

$$G(x) = \inf\left\{\rho^{p_{m/H}} > 0: \left[\left[Sup_{\ k,l} M_1\left(\frac{|\Delta x_{k,l}|}{\rho}\right)\right]^{p_{k,l}}\right]^{1/H} \le 1\right\}$$

$$G(y) = \inf\left\{\rho^{p_{m/H}} > 0: \left[\left[Sup_{\ k,l} M_2\left(\frac{|\Delta y_{k,l}|}{\rho}\right)\right]^{p_{k,l}}\right]^{1/H} \le 1\right\}$$

where $H = \max\left(1, sup_{k,l} p_{k,l}\right)$.

Let U be the set of all double sequence spaces $u = u_{k,l}$, such that $u_{k,l} \neq 0$ $(k = 1, 2, \dots)$, $(l = 1,2, \dots)$

The double sequence spaces in Asma and Colak sense we can defined

$$2l_\infty(u, \Delta, p) = \left\{(x,y) \in \omega : \left(u_{k,l}\Delta x_{k,l},\ u_{k,l}\Delta y_{k,l}\right) \in 2l_\infty(p)\right\}$$

$$2c(u, \Delta, p) = \left\{(x,y) \in \omega : \left(u_{k,l}\Delta x_{k,l}, u_{k,l}\Delta y_{k,l}\right) \in 2c(p)\right\}$$

And

$$2c_\circ(u, \Delta, p) = \left\{(x,y) \in \omega : \left(u_{k,l}\Delta x_{k,l}, u_{k,l}\Delta y_{k,l}\right) \in 2c_\circ(p)\right\}$$

where $\Delta_u x_{k,l} = u_{k,l}x_{k,l} - u_{k,l+1}x_{k,l+1} - u_{k+1,l}x_{k+1,l} + u_{k+1,l+1}x_{k+1,l+1}$

$\Delta_u y_{k,l} = u_{k,l}y_{k,l} - u_{k,l+1}y_{k,l+1} - u_{k+1,l}y_{k+1,l} + u_{k+1,l+1}y_{k+1,l+1}$

$k = 1,2, \dots, l = 1,2, \dots$

The double sequence spaces we define by using Asma sense as the following:

$2l_\infty(u\,,\Delta,M,p) = \left\{(x,y) \in \omega : Sup\ _{k,l}\left[M_1\left(\frac{|u_{k,l}\Delta\, x_{k,l}|}{\rho}\right) \vee\ M_2\left(\frac{|u_{k,l}\Delta\, y_{k,l}|}{\rho}\right)\right]^{p_{k,l}} < \infty,\ for\ some\ \rho > 0\right\}$,

$2c(u\,,\Delta\,,M,p) = \left\{(x,y) \in \omega : \lim_{k,l\to\infty}\left[M_1\left(\frac{|u_{k,l}\Delta\, x_{k,l}-\ell_1|}{\rho}\right) \vee\ M_2\left(\frac{|u_{k,l}\Delta\, y_{k,l}-\ell_2|}{\rho}\right)\right]^{p_{k,l}} = 0,\ for\ some\ \rho\ > 0\ \ell_1,\ell_2\ \in \mathbb{C}\right\}$,

$2c_0\,(u\,,\Delta\,,M,p) = \left\{(x,y) \in \omega : \lim_{k,l\to\infty}\left[M_1\left(\frac{|u_{k,l}\Delta\, x_{k,l}|}{\rho}\right) \vee\ M_2\left(\frac{|u_{k,l}\Delta\, y_{k,l}|}{\rho}\right)\right]^{p_{k,l}} = 0,\ for\ some\ \rho\ > 0\right\}$.

where M_1, M_2 is an Orlicz function , and $u \in U$

In this study, we further generalize these space as follows :

Let $M_1 = M_{1_{k,l}}$, $M_2 = M_{2_{k,l}}$ and $M = M_{k,l} = \left(M_{1_{k,l}}, M_{2_{k,l}}\right)$ be a double sequence of Orlicz functions such that be a double sequence space and n be appositive integer , and using the notation $\Delta_u^n x_{k,l}$ for $u_{k,l}\Delta^n x_{k,l}$ and $\Delta_u^n y_{k,l}$ for $u_{k,l}\Delta^n y_{k,l}$ such that $\left(\Delta_u^n x_{k,l}\,,\Delta_u^n y_{k,l}\right)$ for $\left(u_{k,l}\Delta^n x_{k,l}\,,u_{k,l}\Delta^n y_{k,l}\right)$ we define

$2c_0\left(\Delta_u^n, M_{k,l}, p, s\right) = \left\{(x,y) \in \omega : \lim_{k,l\to\infty}(kl)^{-s}\left[M_{1_{k,l}}\left(\frac{|\Delta_u^n x_{k,l}|}{\rho}\right) \vee\ M_{2_{k,l}}\left(\frac{|\Delta_u^n y_{k,l}|}{\rho}\right)\right]^{p_{k,l}} = 0,\ for\ some\ \rho\ > 0\,, s \geq 0\right\}$

$2c\left(\Delta_u^n, M_{k,l}, p, s\right) = \left\{(x,y) \in \omega : \lim_{k,l\to\infty}(kl)^{-s}\left[M_{1_{k,l}}\left(\frac{|\Delta_u^n x_{k,l}-\ell_1|}{\rho}\right) \vee M_{2_{k,l}}\left(\frac{|\Delta_u^n y_{k,l}-\ell_2|}{\rho}\right)\right]^{p_{k,l}} = 0,\ for\ some\ \rho\ > 0\,, \ell_1,\ell_2 \in \mathbb{C}\,, s \geq 0\right\}$

$2l_\infty(\Delta_u^n, M_{k,l}, p, s) = \left\{(x,y) \in \omega : Sup\ _{k,l}(kl)^{-s}\left[M_{1_{k,l}}\left(\frac{|\Delta_u^n x_{k,l}|}{\rho}\right) \vee M_{2_{k,l}}\left(\frac{|\Delta_u^n y_{k,l}|}{\rho}\right)\right]^{p_{k,l}} < \infty,\ for\ some\ \rho > 0, s \geq 0\right\}$.

Where $\Delta_u^n x_{k,l} = \Delta_u^{n-1} x_{k,l} - \Delta_u^{n-1} x_{k,l+1} - \Delta_u^{n-1} x_{k+1,l} - \Delta_u^{n-1} x_{k+1,l+1}$

$\Delta_v^n y_{k,l} = \Delta_u^{n-1} y_{k,l} - \Delta_u^{n-1} y_{k,l+1} - \Delta_u^{n-1} y_{k+1,l} - \Delta_u^{n-1} y_{k+1,l+1}$

Such that

$\Delta_u^n x_{k,l} = \sum_{r_1=0}^n \sum_{r_2=0}^n (-1)^{r_1+r_2}\binom{n}{r_1}\binom{n}{r_2} u_{k+r_1\,,l+r_2} x_{k+r_1\,,l+r_2}$

$\Delta_u^n y_{k,l} = \sum_{r_1=0}^n \sum_{r_2=0}^n (-1)^{r_1+r_2}\binom{n}{r_1}\binom{n}{r_2} u_{k+r_1\,,l+r_2} y_{k+r_1,l+r_2}$

$\Delta_u^0 x_{k,l} = u_{k,l} x_{k,l}$, $\Delta_u^0 y_{k,l} = u_{k,l} y_{k,l}$

$\Delta_u x_{k,l} = u_{k,l} x_{k,l} - u_{k,l+1} x_{k,l+1} - u_{k+1,l} x_{k+1,l} + u_{k+1,l+1} x_{k+1,l+1}$

$\Delta_u y_{k,l} = u_{k,l} y_{k,l} - u_{k,l+1} y_{k,l+1} - u_{k+1,l} y_{k+1,l} + u_{k+1,l+1} y_{k+1,l+1}$

If $M = M_{k,l}$ such that $M_1 = M_{1_{k,l}}, M_2 = M_{2_{k,l}}$ for all k, l, s = 0 and $n = 1$, then these spaces reduce to those of Asma idea.

It is easy to see that these spaces are paranormed space with $G_{u,n}(x,y) = inf\left\{\rho^{p_m/H} > 0 : \left[sup_{k,l}\left[(kl)^{-s}\left[M_{1_{k,l}}\left(\frac{|\Delta_u^n x_{k,l}|}{\rho}\right) \vee M_{2_{k,l}}\left(\frac{|\Delta_u^n y_{k,l}|}{\rho}\right)\right]\right]^{p_{k,l}}\right]^{1/H} \leq 1\right\}$

Such that

$$G_{u,n}(x) = inf\left\{\rho^{p_m/H} > 0 : \left[sup_{k,l}\left[(kl)^{-s} M_{1_{k,l}}\left(\frac{|\Delta_u^n x_{k,l}|}{\rho}\right)\right]^{p_{k,l}}\right]^{1/H} \leq 1\right\}$$

$$G_{u,n}(y) = inf\left\{\rho^{p_m/H} > 0 : \left[sup_{k,l}\left[(kl)^{-s} M_{2_{k,l}}\left(\frac{|\Delta_u^n y_{k,l}|}{\rho}\right)\right]^{p_{k,l}}\right]^{1/H} \leq 1\right\}$$

where $H = \max\left(1, sup_{k,l} p_{k,l}\right)$

Now, we prove the following theorems

Theorem 2.1.2.4. $l_\infty(\Delta_u^n, M_{k,l}, p, s)$ is complete paranormed space with

$$G_{u,n}(x,y) = inf\left\{\rho^{p_m/H} > 0 : \left[sup_{k,l}\left[(kl)^{-s}\left[M_{1_{k,l}}\left(\frac{|\Delta_u^n x_{k,l}|}{\rho}\right) \vee M_{2_{k,l}}\left(\frac{|\Delta_u^n y_{k,l}|}{\rho}\right)\right]\right]^{p_{k,l}}\right]^{1/H} \leq 1\right\}$$

Proof. Let (x^i, y^i) be any double Cauchy sequence in $2\, l_\infty(\Delta_u^n, M_{k,l}, p, s)$ where x^i, y^i be a Cauchy sequence in $l_\infty\left(\Delta_u^n, M_{1_{k,l}}, p, s\right)$, $l_\infty\left(\Delta_u^n, M_{2_{k,l}}, p, s\right)$ respectively. let $r, x_0, y_0 > 0$ be a fixed ,then for each $\frac{\epsilon}{rx_0} > 0$, $\frac{\epsilon}{r\,y_0} > 0$ we have $\left(\frac{\epsilon}{rx_0}, \frac{\epsilon}{r\,y_0}\right) > (0,0)$ there exists a positive integer L such that $G_{u,n}(x^i - x^j) < \frac{\epsilon}{rx_0}$ and $G_{u,n}(y^i - y^j) < \frac{\epsilon}{r\,y_0}$, we have $G_{u,n}(x^i - x^j, y^i - y^j) < \left(\frac{\epsilon}{rx_0}, \frac{\epsilon}{ry_0}\right)$, for all $i, j \geq L$

Using the definition of paranormed ,we have

$$\left[Sup_{k,l}\,(kl)^{-s}\left[M_{1_{k,l}}\left(\frac{\left|\Delta_u^n x_{k,l}^i - \Delta_u^n x_{k,l}^j\right|}{G_{u,n}(x^i - x^j)}\right) \vee M_{2_{k,l}}\left(\frac{\left|\Delta_u^n y_{k,l}^i - \Delta_u^n y_{k,l}^j\right|}{G_{u,n}(y^i - y^j)}\right)\right]^{p_{k,l}}\right]^{1/H} \leq 1, \text{ for all } i, j \geq L$$

Thus

$$\left[Sup_{k,l}\,(kl)^{-s}\left[M_{1_{k,l}}\left(\frac{\left|(\Delta_u^n x_{k,l}^i - \Delta_u^n x_{k,l}^j)\right|}{G_{u,n}(x^i - x^j)}\right) \vee M_{2_{k,l}}\left(\frac{\left|\Delta_u^n y_{k,l}^i - \Delta_u^n y_{k,l}^j\right|}{G_{u,n}(y^i - y^j)}\right)\right]\right]^{p_{k,l}} \leq 1, \text{ for all } i, j \geq L$$

Therefore $(kl)^{-s}\left[M_{1_{k,l}}\left(\frac{\left|\Delta_u^n x_{k,l}^i - \Delta_u^n x_{k,l}^j\right|}{G_{u,n}(x^i - x^j)}\right) \vee M_{2_{k,l}}\left(\frac{\left|\Delta_u^n y_{k,l}^i - \Delta_u^n y_{k,l}^j\right|}{G_{u,n}(y^i - y^j)}\right)\right] \leq 1$, for each $k, l \geq 0$ and for all $i, j \geq L$. For $r > 0$ with $(kl)^{-s}\left[M_{1_{k,l}}\left(\frac{rx_0}{2}\right) \vee M_{2_{k,l}}\left(\frac{ry_0}{2}\right)\right] \geq 1$, we have

$$(kl)^{-s}\left[M_{1_{k,l}}\left(\frac{\left|\Delta_u^n x^i_{k,l}-\Delta_u^n x^j_{k,l}\right|}{G_{u,n}(x^i-x^j)}\right)\vee M_{2_{k,l}}\left(\frac{\left|\Delta_u^n y^i_{k,l}-\Delta_u^n y^j_{k,l}\right|}{G_{u,n}(y^i-y^j)}\right)\right] \le (kl)^{-s}\left[M_{1_{k,l}}\left(\frac{rx_0}{2}\right)\vee M_{2_{k,l}}\left(\frac{r\,y_0}{2}\right)\right]. ..$$

This implies that

$\left|\Delta_u^n x^i_{k,l}-\Delta_u^n x^j_{k,l}\right| \le \frac{rx_0}{2}.\frac{\epsilon}{rx_0} = \frac{\epsilon}{2}$, $\left|\Delta_u^n y^i_{k,l}-\Delta_u^n y^j_{k,l}\right| \le \frac{r\,y_0}{2}.\frac{\epsilon}{ry_0} = \frac{\epsilon}{2}$ we have

$$\left|\left(\Delta_u^n x^i_{k,l}-\Delta_u^n x^j_{k,l}\,,\,\Delta_u^n y^i_{k,l}-\Delta_u^n y^j_{k,l}\right)\right| \le \left(\frac{rx_0}{2},\frac{ry_0}{2}\right).\left(\frac{\epsilon}{rx_0},\frac{\epsilon}{ry_0}\right) = \left(\frac{\epsilon}{2},\frac{\epsilon}{2}\right).$$

Since $u_{kl}\neq 0$ for all k,l , we get that $\left|\Delta^n x^i_{k,l}-\Delta^n x^j_{k,l}\right| \le \frac{\epsilon}{2}$, $\left|\Delta^n y^i_{k,l}-\Delta^n y^j_{k,l}\right| \le \frac{\epsilon}{2}$ we have $\left|\left(\Delta^n x^i_{k,l}-\Delta^n x^j_{k,l}\,,\,\Delta^n y^i_{k,l}-\Delta^n y^j_{k,l}\right)\right| \le \left(\frac{\epsilon}{2},\frac{\epsilon}{2}\right)$, for all $i,j\ge L$. Hence $\left(\Delta^n x^i_{k,l},\Delta^n y^i_{k,l}\right)$ is a double Cauchy sequence in $\mathbb{R}\times\mathbb{R}$. Therefore for each ϵ $(0<\epsilon<1)$, there exists a positive integers L such that $\left|\Delta^n x^i_{k,l}-\Delta^n x^j_{k,l}\right|<\epsilon$, $\left|\Delta^n y^i_{k,l}-\Delta^n y^j_{k,l}\right|<\epsilon$ we have $\left|\left(\Delta^n x^i_{k,l}-\Delta^n x^j_{k,l}\,,\,\Delta^n y^i_{k,l}-\Delta^n y^j_{k,l}\right)\right|<(\epsilon,\epsilon)$, for all $i,j\ge L.$

Now, using the continuity of $M_{1_{k,l}}$ $M_{2_{k,l}}$ for each k,l ,we get that

$$\left[sup_{k,l\ge L}\left[(kl)^{-s}\left[M_{1_{k,l}}\left(\frac{\left|\Delta_u^n x^i_{k,l}-\lim_{j\to\infty}\Delta_u^n x^j_{k,l}\right|}{\rho}\right)\vee M_{2_{k,l}}\left(\frac{\left|\Delta_u^n y^i_{k,l}-\lim_{j\to\infty}\Delta_u^n y^j_{k,l}\right|}{\rho}\right)\right]\right]^{p_{k,l}}\right]^{1/H} \le 1$$

Thus

$$\left[sup_{k,l\ge L}\left[(kl)^{-s}\left[M_{1_{k,l}}\left(\frac{\left|\Delta_u^n x^i_{k,l}-\Delta_u^n x_{k,l}\right|}{\rho}\right)\vee M_{2_{k,l}}\left(\frac{\left|\Delta_u^n y^i_{k,l}-\Delta_u^n y_{k,l}\right|}{\rho}\right)\right]\right]^{p_{k,l}}\right]^{1/H} \le 1$$

Take infimum of such ρ's we have

$$\inf\left\{\rho^{p_m/H}>0:\left[sup_{k,l\ge L}\left[(kl)^{-s}\left[M_{1_{k,l}}\left(\frac{\left|\Delta_u^n x^i_{k,l}-\Delta_u^n x_{k,l}\right|}{\rho}\right)\vee M_{2_{k,l}}\left(\frac{\left|\Delta_v^n y^i_{k,l}-\Delta_v^n y_{k,l}\right|}{\rho}\right)\right]\right]^{p_{k,l}}\right]^{1/H_1}\le 1\right\}\le$$

ϵ , for all $i\ge L$ and $j\to\infty$

Since $(x^i,y^i)\in 2l_\infty(\Delta_u^n,M_{kl},p,s)$, and M_{kl} is an Orlicz function , $M_{1_{k,l}}$, $M_{2_{k,l}}$ is in Orlicz function, for each k,l and therefore continuous, we get that $(x,y)\in 2l_\infty(\Delta_u^n,M_{kl},p,s)$ ◙

Theorem 2.1.2.5. Let $0\le p_{k,l}\le q_{k,l}<\infty$, and for all k,l . Then $2c_0(\Delta_u^n,M_{k,l},p\,,s)\subseteq 2c_0(\Delta_u^n,M_{k,l},q\,,s)$

Proof. Let $(x,y)\in 2c_0(\Delta_u^n,M_{k,l},p\,,s)$. Then there exists some $\rho>0$ such that

$$\lim_{k,l\to\infty}(kl)^{-s}\left[M_{1_{k,l}}\left(\frac{|\Delta_u^n x_{k,l}|}{\rho}\right)\vee M_{2_{k,l}}\left(\frac{|\Delta_u^n y_{k,l}|}{\rho}\right)\right]^{p_{k,l}} = 0$$

This implies that

$$(kl)^{-s}\left[M_{1_{k,l}}\left(\frac{|\Delta_u^n x_{k,l}|}{\rho}\right)\vee M_{2_{k,l}}\left(\frac{|\Delta_u^n y_{k,l}|}{\rho}\right)\right]^{p_{k,l}}\leq 1,$$

for sufficiently large k,l since $M_{1_{k,l}}$, $M_{2_{k,l}}$ is anon decreasing for each k,l . Hence

$$\lim_{k,l\to\infty}\left[(kl)^{-s}\left[M_{1_{k,l}}\left(\frac{|\Delta_u^n x_{k,l}|}{\rho}\right)\vee M_{2_{k,l}}\left(\frac{|\Delta_u^n y_{k,l}|}{\rho}\right)\right]\right]^{q_{k,l}}\leq$$

$$\lim_{k,l\to\infty}\left[(kl)^{-s}\left[M_{1_{k,l}}\left(\frac{|\Delta_u^n x_{k,l}|}{\rho}\right)\vee M_{2_{k,l}}\left(\frac{|\Delta_u^n y_{k,l}|}{\rho}\right)\right]\right]^{p_{k,l}}=0$$

That is $(x,y)\in 2c_0(\Delta_u^n, M_{k,l}, q, s)$◙

Theorem 2.1.2.6. i) Let $0<\inf p_{k,l}\leq p_{k,l}\leq 1$. Then $2\,c_0(\Delta_u^n, M_{k,l}, p, s)\subseteq 2c_0(\Delta_u^n, M_{k,l}, s)$

ii) Let $1\leq p_{k,l}\leq \sup p_{k,l}<\infty$. Then, $2c_0(\Delta_u^n, M_{k,l}, s)\subseteq 2c_0(\Delta_u^n, M_{k,l}, p, s)$

proof. Let $(x,y)\in 2c_0(\Delta_u^n, M_{k,l}, p, s)$. Then $\lim_{k,l\to\infty}\left[(kl)^{-s}\left[M_{1_{k,l}}\left(\frac{|\Delta_u^n x_{k,l}|}{\rho}\right)\vee M_{2_{k,l}}\left(\frac{|\Delta_u^n y_{k,l}|}{\rho}\right)\right]\right]^{p_{k,l}}=0$

Since $0<\inf p_{k,l}\leq p_{k,l}\leq 1$, we have

$$\lim_{k,l\to\infty}(kl)^{-s}\left[M_{1_{k,l}}\left(\frac{|\Delta_u^n x_{k,l}|}{\rho}\right)\vee M_{2_{k,l}}\left(\frac{|\Delta_u^n y_{k,l}|}{\rho}\right)\right]\leq$$

$$\lim_{k,l\to\infty}\left[\left[(kl)^{-s}\left[M_{1_{k,l}}\left(\frac{|\Delta_u^n x_{k,l}|}{\rho}\right)\vee M_{2_{k,l}}\left(\frac{|\Delta_u^n y_{k,l}|}{\rho}\right)\right]\right]^{p_{k,l}}\right]=0$$

That is $(x,y)\in 2c_0(\Delta_u^n, M_{k,l}, s)$

ii) $1\leq p_{k,l}$ for each k,l and sup $p_{k,l}<\infty$. let $(x,y)\in 2\,c_0(\Delta_u^n, M_{k,l}, s)$, then for each ϵ $(0<\epsilon<1)$, there exists a positive integer L such that

$$(kl)^{-s}\left[M_{1_{k,l}}\left(\frac{|\Delta_u^n x_{k,l}|}{\rho}\right)\vee M_{2_{k,l}}\left(\frac{|\Delta_u^n y_{k,l}|}{\rho}\right)\right]\leq\epsilon, \text{ for all } k,l\geq L$$

Since $1\leq p_{k,l}\leq \sup p_{k,l}<\infty$, we have

$$\lim_{k,l\to\infty}\left[(kl)^{-s}\left[M_{1_{k,l}}\left(\frac{|\Delta_u^n x_{k,l}|}{\rho}\right)\vee M_{2_{k,l}}\left(\frac{|\Delta_u^n y_{k,l}|}{\rho}\right)\right]\right]^{p_{k,l}}\leq \lim_{k,l\to\infty}(kl)^{-s}\left[M_{1_{k,l}}\left(\frac{|\Delta_u^n x_{k,l}|}{\rho}\right)\vee M_{2_{k,l}}\left(\frac{|\Delta_u^n y_{k,l}|}{\rho}\right)\right]\leq \epsilon<1$$

$(x,y)\in 2c_0(\Delta_u^n, M_{k,l}, p, s)$ ◙